LOWER BOUNDS ON NODAL SETS OF EIGENFUNCTIONS VIA THE HEAT FLOW

STEFAN STEINERBERGER

Abstract. We study the size of nodal sets of Laplacian eigenfunctions on compact Riemannian manifolds without boundary and recover the currently optimal lower bound by comparing the heat flow of the eigenfunction with that of an artifically constructed diffusion process. The same method should apply to a number of other questions; for example, we prove a sharp result saying that a nodal domain cannot be entirely contained in a small neighbourhood of a 'reasonably flat' surface. We expect the arising concepts to have more connections to classical theory and pose some conjectures in that direction.

1. Introduction

We consider a compact $n-$dimensional $C^\infty-$manifold (M,g) without boundary. Writing Δ_g for the Laplace-Beltrami operator, we are interested in Laplacian eigenfunctions

$$-\Delta_g u = \lambda u.$$

A natural object of study is the measure of the nodal set

$$Z = \{x \in M : u(x) = 0\}.$$

An old conjecture of Yau [22] states that $|Z| \sim \lambda^{\frac{1}{2}}$. For real-analytic (M,g) this was proven by Donnelly & Fefferman [7], however, in the generic $C^\infty-$case even $n = 2$ is still open. In two dimensions, the best bounds are

$$\lambda^{\frac{1}{2}} \lesssim |Z| \lesssim \lambda^{\frac{3}{4}},$$

where the lower bound is by Brüning [3] and Yau (unpublished), independently, and the upper bound is due to Donnelly & Fefferman [8] (with another proof given by Dong [6]). After exponentially decaying lower bounds were given by Hardt & Simon [11], the first polynomial bound is due to Sogge & Zelditch [18]

$$|Z| \gtrsim \lambda^{\frac{7-3n}{8}},$$

although, as was later pointed out, an earlier result by Mangoubi [15] can be combined with the isoperimetric inequality to yield

$$|Z| \gtrsim \lambda^{\frac{3-n}{2}-\frac{1}{2n}}.$$

Currently, the best bound is the following.

Theorem. *The volume of nodal sets satisfies*

$$\mathcal{H}^{n-1}\left(\{x \in M : u(x) = 0\}\right) \gtrsim \lambda^{\frac{3-n}{4}}.$$

This was first proven by Colding & Minicozzi [5]. Subsequently, different proofs were given by Hezari & Sogge [11], Hezari & Wang [12] (for $n \leq 5$) and Sogge & Zelditch [19]. The arguments tend to be either local estimates on small balls in the style of Donnelly-Fefferman or global integral formulae. It is the purpose of this paper to give a new local approach exploiting the fact that a Laplacian eigenfunction behaves nicely under the heat flow. The approach is fully self-contained with the exception of our using a global inequality due to Sogge & Zelditch [19]

$$\lambda \frac{\|u\|_{L^1(M)}}{\|\nabla u\|_{L^\infty(M)}} \gtrsim \lambda^{\frac{3-n}{4}},$$

which is known to be sharp on spherical harmonics. We also sketch a variant of our proof that comes to rely on $\lambda^{\frac{1}{2}}\|u\|_{L^1(M)} \gtrsim \lambda^{\frac{3-n}{4}}\|u\|_{L^\infty(M)}$ (also used by Sogge & Zelditch [18]), which is

easily seen to be equivalent because of $\|\nabla u\|_{L^{\infty}(M)} \sim \lambda^{1/2}\|u\|_{L^{\infty}(M)}$.

As a by-product we show that, for c sufficiently small, a nodal domain cannot be contain in a $c\lambda^{-1/2}-$neighbourhood of a sufficiently flat $(n-1)-$dimensional surface in M. The statement is easily seen to be sharp because there are eigenfunctions on the flat torus $\mathbb{T}^n$ such that any nodal domains is contained in a $C\lambda^{-1/2}-$neighbourhood of a geodesic of length 1. As another by-product we formulate two very hard conjectures in the spirit of our approach whose resolution would imply a slightly sharper version of Yau's conjecture.

2. Statement of results

2.1. **The main idea.** The main idea is as follows: the heat equation with a Laplacian eigenfunction as initial data and Dirichlet conditions on the nodal set has the explicit solution $e^{-\lambda t}u(x)$ (this is also the solution of the heat flow without any boundary conditions, however, we will be working locally). In particular, we have precise control on the rate of decay of the L^1-norm in time. A natural candidate for comparison is the heat equation with the same initial data but Neumann conditions on the nodal set, which conserves L^1. However, the entire difference between Dirichlet and Neumann heat flow is caused by the nodal set and if it was too small, it couldn't account for the difference in behavior.

Our proof is not actually using the Neumann solution because it requires some regularity on the boundary and would necessitate using reflected Brownian motion whose construction is nontrivial around the singular set. Instead, we choose another way and construct a stochastic process which might be interesting in itself: for small times it acts as a diffusion but as time grows the process converges back to initial data. We don't expect any serious obstacles if one were to work with actual Neumann solutions, some remarks on how to proceed are sketched in the end of the paper.

Theorem 1. *We have*

$$\mathcal{H}^{n-1}\left(\{x \in M : u(x) = 0\}\right) \gtrsim \lambda^{\frac{3-n}{4}}.$$

The (purely local) proof will give a sum consisting of local terms over all nodal domains D, the lower bound is then implied by a global inequality due to Sogge & Zelditch [19]

$$\lambda \sum_{D} \frac{\|u\|_{L^1(D)}}{\|\nabla u\|_{L^{\infty}(D)}} \geq \lambda \frac{\|u\|_{L^1(M)}}{\|\nabla u\|_{L^{\infty}(M)}} \gtrsim \lambda^{\frac{3-n}{4}}.$$

2.2. **Geometry of nodal sets.** We expect the argument to be applicable to more general diffusion processes and possibly other questions about Laplacian eigenfunctions. One example is the shape of nodal domains, where we briefly describe a simple geometrical result. It deals with the question whether nodal domains can be contained in a small neighbourhood of a 'flat' surface of codimension 1. In two dimensions, the statement reduces to a statement mentioned by Mangoubi [16] and to a theorem of Hayman [10], however, in this generality it seems to be new.

Let $\Sigma \subset M$ be an arbitrary smooth $(n-1)-$dimensional surface. We ask whether a nodal domain can be contained in a small neighbourhood of Σ. The $\varepsilon-$neighbourhood of a generic geodesic (being itself as 'flat' as possible) on the torus $\mathbb{T}^2$ already coincides with the entire torus - we therefore need to place some restrictions on Σ for the question to be meaningful. Using $d_g(\cdot,\cdot)$ to denote the geodesic distance, we call Σ admissible up to distance r if

(1) $\forall x \in M : d_g(x,\Sigma) \leq r \implies \#\{y \in \Sigma : d(x,y) = d(x,\Sigma)\} = 1$
(2) and, additionally, at any point $x \in \Sigma$ the surface can be locally written as the graph of a function ϕ from the tangent plane in x to $\mathbb{R}$ with the spectral norm of the Hessian satisfying

$$\|H\phi\|_2 \leq \lambda.$$

The first condition precludes the geodesics scenario while the second condition imposes local regularity on the surfaces. Both conditions could arguably be weakened, however, we are interested in the geometric implications of a 'smooth' theorem rather than optimal low regularity.

Theorem 2. *There is a constant $c > 0$ depending only on (M, g) such that if $\Sigma \subset M$ is admissible up to distance $\lambda^{-1/2}$, then no nodal domain can be a subset of the $c\lambda^{-1/2}-$neighbourhood of Σ.*

As mentioned above, the function $u(x) = \operatorname{Re}\exp(i\sqrt{\lambda}x)$ on $\mathbb{T}^2$ endowed with the flat metric has all its nodal domains contained in a $0.5\lambda^{-1/2}$ neighbourhood of a geodesic of length 1 (being admissible up to $r = 0.5$) and the example easily generalizes to higher dimensions.

It is not difficult to see (from the proof) that the statement can be generalized to the union of admissible sets assuming they are sufficiently transversal at points of intersection and assuming the complement of the union does not have small connected components. Indeed, the proof immediately carries over to the following classical theorem (where the inradius is defined as the radius of the largest ball fully contained in the domain).

Theorem (Hayman, [10]). *There exists a constant $c \geq 900^{-1}$ such that for any simply connected domain $\Omega \subset \mathbb{R}^2$ with inradius ρ*

$$\lambda_1(\Omega) \geq \frac{c}{\rho^2}.$$

Note that the assumption of being simply connected is crucial but can be relaxed provided the 'holes' aren't too small - this is the case for Laplacian eigenfunctions, where the lack of simple connectivity comes from other nodal domains, which cannot be too small themselves.

2.3. **Open questions.** The main idea of getting information on the nodal set by constructing diffusion processes which deviate in behavior at the boundary – because its infinitesimally generated particles are absorbed/reflected differently – relies on having the right concepts. As it turns out, a rather crucial concept is the notion of heat content, which seems to have been studied by many people but in a predominantly asymptotic sense (see, e.g. [2]). We conjecture an isoperimetric principle for the heat content and show how such a principle would imply yet another proof of the main theorem. It also allows for a natural refinement of Yau's conjecture and seems to be the suitable for proving a Lieb-type generalization of Hayman's theorem – this is discussed in the last section of the paper.

As for notation, the symbols $\lesssim$ and $\sim$ will always denote absolute constants depending only on the manifold (M, g).

3. Proof of the main Theorem.

3.1. **Heat content.** Given an open subset $N \subset M$, we use $p_t(x)$ to denote the solution to the following heat equation

$$\begin{aligned} (\partial_t - \Delta_g)p_t(x) &= 0 \qquad && x \in N \\ p_t(x) &= 1 && x \in \partial N \\ p_0(x) &= 0 && x \in N. \end{aligned}$$

The Feynman-Kac formula implies that this can be understood as the probability that a Brownian motion particle started in x will hit the boundary within t units of time. The quantity

$$\int_N p_t(x)dx$$

is called the heat content of N at time t. It can be seen as a 'soft' measure of boundary size – for large times the function will be roughly of size 1 in the entire domain and all information on the size of the boundary will be lost. However, within t units of time a typical Brownian motion particle travels a distance of $\sim t^{1/2}$. This can be immediately seen with Varadhan's large deviation formula [21]

$$\lim_{t \to 0} -4t \log K(t, x, y) = d(x, y)^2,$$

where $K(\cdot, \cdot, \cdot)$ is the heat kernel on the manifold (M, g). Hence, for very small times t, the function $p_t(x)$ should therefore be mostly supported around the boundary and this yields a connection to the size of the boundary.

3.2. **Definitions.** Let D be an arbitrary nodal domain. Without loss of generality, we assume the eigenfunction $u(x)$ to be positive within D: otherwise consider $-u(x)$. Given $u(x)$, we define a one-parameter functions $v(t,x)$ as solutions to the heat equation with $u(x)\big|_D$ as initial data and Dirichlet conditions. We set

$$v(t,x) := e^{-\lambda t}u(x)$$

and note that $v(t,x)$ then solves

$$\begin{aligned}(\partial_t - \Delta_g)v(t,x) &= 0 \quad \text{on } D \setminus \{u(x)=0\}\\ v(t,x) &= 0 \quad \text{on } \{u(x)=0\}\\ v(0,x) &= u(x) \quad \text{on } D.\end{aligned}$$

The Feynman-Kac formula for the Dirichlet problem is classical (see e.g. Taylor [20]). Given an open domain $\Omega \in \mathbb{R}^n$, $f \in C_0^\infty(\Omega)$, $x \in \Omega$, then

$$(e^{t\Delta_D}f)(x) = \mathbb{E}_x(f(\omega(t))\psi_\Omega(\omega,t)),$$

where $t > 0$ is arbitrary, $\omega(t)$ denotes an element of the probability space of Brownian motions starting in x, $\mathbb{E}_x$ is to be understood with regards to the measure of that probability space and

$$\psi_\Omega(\omega,t) = \begin{cases} 1 & \text{if } \omega([0,t]) \subset \Omega \\ 0 & \text{otherwise.}\end{cases}$$

Here we see the connection with the heat content: for any point $x \in \Omega$ and any $t > 0$

$$\mathbb{E}_x(\psi_\Omega(\omega,t)) = 1 - p_t(x).$$

For reasons that will become apparent in the proof, for $f \in C_0^\infty(\Omega)$ we define a second 'diffusion' operator Ξ via

$$(e^{t\Xi}f)(x) := \mathbb{E}_x(f(\omega(t))\psi_\Omega(\omega,t)) + \mathbb{E}_x(1-\psi_\Omega(\omega,t))f(x).$$

This operator is initially smoothing but ceases being so as time progresses. Indeed, if $\Omega \subset M$ such that $M \setminus \Omega$ is open, then if Ξ is adapted to Ω it is easy to see that for every $x \in \Omega$

$$\lim_{t\to\infty}(e^{t\Xi}f)(x) = f(x).$$

Finally, we claim that

$$\int_\Omega e^{t\Xi}f dx = \int_\Omega f dx,$$

which is equivalent to

$$\int_\Omega p_t(x)u(x)dx = \int_\Omega (1-e^{t\Delta_D})u(x)dx.$$

A stochastic argument would be to say that among paths not leaving the domain, it is equally likely to start in a point x and end in a point y than the other way around – a statement that follows from the symmetry $K(t,x,y) = K(t,y,x)$ of the heat kernel.

3.3. **A Comparison Lemma.** We are interested in comparing the behavior of the Dirichlet solution $e^{t\Delta_D}u$ with the behavior of $e^{t\Xi}u$ on a fixed nodal domain D, where we assume without loss of generality that $u\big|_D \geq 0$ (otherwise: consider $-u(x)$). It is obvious from the definition that

$$e^{t\Xi}u \geq e^{t\Delta_D}u.$$

Lemma. *There exists a constant $C > 0$ depending only on (M,g) such that*

$$e^{t\Xi}u - e^{t\Delta_D}u \leq Ct^{1/2}p_t(x)\|\nabla u\|_{L^\infty}.$$

Proof. The definition yields that

$$e^{t\Xi}u - e^{t\Delta_D}u = p_t(x)u(x).$$

The quantity $p_t(x)$ is localized at a $t^{1/2}-$neighbourhood of the origin and has an exponentially decaying tail at larger distance. The statement then follows from

$$u(x) \leq d(x,\partial D)\|\nabla u\|_{L^\infty}.$$

□

3.4. **Conclusion.** Fix again an arbitrary nodal domain D and we assume again without loss of generality that $u(x)\big|_D \geq 0$. The heat equation, the comparison lemma and the L^1-conservation of Ξ give

$$\begin{aligned} e^{-\lambda t}\int_D u(x)dx &= \int_D e^{t\Delta_D}u(x)dx \\ &\geq \int_D e^{t\Xi}u(x)dx - Ct^{1/2}\|\nabla u\|_{L^\infty}\int_D p_t(x)dx \\ &= \int_D u(x)dx - Ct^{1/2}\|\nabla u\|_{L^\infty}\int_D p_t(x)dx \end{aligned}$$

Therefore

$$\int_D p_t(x)dx \gtrsim \frac{1-e^{-\lambda t}}{t^{1/2}}\frac{\|u\|_{L^1(D)}}{\|\nabla u\|_{L^\infty(D)}}.$$

The result now follows from showing that up to constants depending on the manifold

$$\lim_{t\rightarrow 0^+}\frac{1}{\sqrt{t}}\int_D p_t(x)dx \sim \mathcal{H}^{n-1}(\partial D).$$

The involved quantities are essentially local and so is our argument: it is known that the critical set

$$\{x\in D : u(x) = |\nabla u(x)| = 0\}$$

has $(n-1)-$dimensional *Minkowski* measure 0 - recent results by Cheeger, Naber & Valtorta [4] even give bounds on its $(n-2)-$dimensional Minkowski measure. Fix a small $t>0$ and cover the ∂D with cubes of side length $t^{-1/2}$. We call a cube *regular* if it does not contain an element of the singular set and *singular* otherwise. From the result above, it follows that the number of singular cubes is of order $o(t^{(-(n-1)/2)})$.

Regular cubes. Let $t>0$ be fixed and let Q be a regular cube. Once a cube is regular, there is no need to further refine it as time tends to 0. To see this, we fix a second time-parameter $z>0$ and study the problem for $z\rightarrow 0^+$ locally in Q. Our desired result follows easily from Varadhan's large deviation formula [21]

$$\lim_{z\rightarrow 0^+} -4z\log K(z,x,y) = d(x,y)^2,$$

where $K(\cdot,\cdot,\cdot)$ is the heat kernel on the manifold (M,g). In particular, it implies that the bulk of the Green function is concentrated in a $\sim z^{1/2}-$neighbourhood of the boundary $\partial D\cap Q$ and using the regularity of the boundary, it is possible to compare hitting probabilities with transition probabilities from D to D^c. This implies the statement.

Singular cubes. It remains to show that the error introduced by those cubes containing an element of the singular set is small: it is not enough to note that their relative proportion is small because they are weighted with a factor $t^{-1/2}$, which becomes singular for small times. Using $0\leq p_t(x)\leq 1$ gives

$$\left|\frac{1}{\sqrt{t}}\int_{D_{\text{sing}}} p_t(x)dx\right| \leq \frac{1}{\sqrt{t}}|D_{\text{sing}}|.$$

D_{sing} consists of $o(t^{(-(n-1)/2)})$ cubes of side-length $t^{1/2}$, therefore

$$\frac{1}{\sqrt{t}}|D_{\text{sing}}| \leq \frac{1}{\sqrt{t}}t^{n/2}o(t^{(-(n-1)/2)}) = o(1).$$

Improved estimates on the Minkowski dimension actually imply a faster rate of decay but these are not necessary for the conclusion of the argument. $\square$

4. Thin nodal sets

In this section, we give a proof of Theorem 2. It is based on using the fact that at scale $\sim \lambda^{-1/2}$ the neighbourhood of a surface admissible up to $\lambda^{-1/2}$ behaves like the neighbourhood of a hyperplane in $\mathbb{R}^n$, which allows for problems to be reduced to well-known one-dimensional facts.

Proof of Theorem 2. We consider the heat flow with Dirichlet conditions on the nodal domain D

$$\begin{aligned} (\partial_t - \Delta_g)v(t,x) &= 0 \quad \text{on } D \setminus \{u(x)=0\} \\ v(t,x) &= 0 \quad \text{on } \{u(x)=0\} \\ v(0,x) &= u(x). \end{aligned}$$

The other case being identical, we assume $u(x)$ to be positive in D. We start by proving

$$\forall\ t>0 \qquad \inf_{x\in D} p_t(x) \leq 1 - e^{-\lambda t}.$$

We argue by contradiction to prove the following slightly stronger statement

$$\forall x \in D \qquad u(x) = \|u\|_{L^\infty(D)} \implies p_t(x) \leq 1-e^{-\lambda t}.$$

Given such a $x \in D$, we see using the heat equation and Feynmann-Kac

$$\begin{aligned} e^{-\lambda t}\|u\|_{L^\infty(D)} &= e^{-\lambda t}u(x) = \mathbb{E}_x(\omega(t)\psi_D(\omega(t))) \\ &\leq \|u\|_{L^\infty(D)}\mathbb{E}_x(\psi_D(\omega(t))) = \|u\|_{L^\infty(D)}(1-p_t(x)). \end{aligned}$$

We now set the time to be $t = \lambda^{-1}$. It remains to show that choosing c small enough, we can derive a contradiction to this bound on $p_t(x)$. Take a small $c>0$ and a $c\lambda^{-1/2}-$neighbourhood of the admissible surface Σ. Assume D to be a nodal domain fully contained in that set and let $x \in D$ be such that $u(x) = \|u\|_{L^\infty(D)}$. The statement we need to contradict is $p_t(x) \leq 1 - e^{-1}$ and we will do so but suitably bounding the probability of leaving the $c\lambda^{-1/2}-$neighbourhood of Σ from below to achieve a contradiction.

Since we are neither interested in the precise dependence of c on the constants in the regularity assumptions (which are set to 1 in the assumptions) nor in getting a good numerical value for c given these assumptions, it suffices to show that we are living at the right scale. From the regularity assumptions on Σ, we see that for a Brownian motion $\omega(t)$ starting in x the function $\mathrm{dist}(x,\omega(t))$ behaves with respect to the probability of being contained in an interval *on these scales* comparable to a one-dimensional Brownian motion. Using $B(t)$ to denote the 1-dimensional Brownian motion on $\mathbb{R}$ the relevant quantity is well-understood and the reflection principle (see e.g. [13]) implies

$$\mathbb{P}\left(\sup_{0<s<\lambda^{-1}} B(t) > c\lambda^{-1/2}\right) = 2\mathbb{P}\left(B(\lambda^{-1}) > c\lambda^{-1/2}\right).$$

However, $B(\lambda^{-1})$ is just a random variable following a normal distribution with mean $\mu = 0$ variance $\sigma = \lambda^{-1/2}$, the contradiction follows from taking c sufficiently small. $\square$

5. Comments and conjectures

We believe the heat content to be a possibly valuable tool in the further study of Laplacian eigenfunctions; this section studies some further implications, in particular we conjecture an isoperimetric statement, which would imply another proof of our main theorem.

5.1. **Heat content isoperimetry.** The heat content is a very stable notion and well-defined even for very rough domains. We consider the following statement to be highly plausible.

Conjecture. *Let (M,g) be a compact $C^\infty-$manifold without boundary. There exists a constant $c>0$ depending only on (M,g) such that for any open subset $N \subset M$ and all times $t>0$*

$$\int_N p_t(x)dx \leq c\mathcal{H}^{n-1}(\partial N)\sqrt{t},$$

where the Hausdorff measure is understood to be ∞ if undefined.

Remarks.

(1) Extremizers of the inequality need to have a smooth boundary: small irregularities in the boundary increase the surface measure but have very limited impact on the left-hand side. It would be interesting to understand the relation between the nature of extremizers and geometric properties of the manifold. Is there a connection to Cheeger sets?

(2) If the domain N has the property that there is a real number $r > 0$ such that each point $x \in N$ is contained in a ball of radius r (possibly centered around another point), then the two quantities should be comparable up to $t \sim r^2$. If N is a nodal domain of the Laplacian, the Faber-Krahn inequality implies that the inradius is at most $\sim \lambda^{-1/2}$ and therefore $t \sim \lambda^{-1}$ is the maximum time up to which we expect the quantities to be comparable.

In particular, we conjecture that for a nodal domain both quantities are indeed comparable up to $t = \lambda^{-1}$. If this could be shown, it would immediately imply

$$\mathcal{H}^{n-1}\left(x \in M : u(x) = 0\right) \lesssim \lambda^{\frac{1}{2}}.$$

5.2. **Isoperimetry implies the main statement.** Assuming the conjecture to be true, the second remark suggests that the maximum viable time for its application to a nodal domain without loss is given by $t = \lambda^{-1}$. We ignore possible issues arising in its construction (see, e.g. Bass & Hsu [1]) and assume the existence of the reflected Brownian motion on the nodal domain.

Theorem 3. *Assuming heat content isoperimetry and existence of reflected Brownian motion, we have*

$$\mathcal{H}^{n-1}\left(\{x \in M : u(x) = 0\}\right) \gtrsim \lambda^{\frac{1}{2}} \sum_{D} \frac{\|u\|_{L^1(D)}}{\|u\|_{L^\infty(D)}} \gtrsim \lambda^{\frac{3-n}{4}},$$

where the sum ranges over all nodal domains D.

We sum the local estimates (using [18])

$$\lambda^{\frac{1}{2}} \frac{\|u\|_{L^1(M)}}{\|u\|_{L^\infty(M)}} \gtrsim \lambda^{\frac{3-n}{4}}.$$

Proof. The proof has strong similarities to our previous argument. Again, without loss of generality, we assume $u(x) > 0$ on D and write $e^{t\Delta_D}$ and $e^{t\Delta_N}$ for evolution under Dirichlet and Neumann data, respectively. Our new comparison estimate is even simpler and states that on a nodal domain D

$$e^{t\Delta_N}u - e^{t\Delta_D}u \leq p_t(x)\|u\|_{L^\infty(D)}.$$

The proof for this comparison statement is easy to sketch: the difference between Dirichlet and Neumann solutions arises from those Brownian motions hitting the boundary. The difference is maximized if all those particles hitting the boundary arrive in a maximum of u after having been reflected.

Integrating the comparison at time $t = \lambda^{-1}$ yields

$$\begin{aligned} e^{-1}\int_D u(x)dx &= \int_D e^{\Delta_D}u(x)dx \\ &\geq \int_D e^{\Delta_N}u(x) - p_t(x)\|u\|_{L^\infty(D)}dx \\ &= \int_D u(x)dx - \|u\|_{L^\infty(D)}\int_D p_t(x)dx. \end{aligned}$$

Therefore, assuming heat content isoperimetry and using the Sogge-Zelditch inequality

$$\mathcal{H}^{n-1}(\partial D) \gtrsim \lambda^{1/2}\frac{\|u\|_{L^1}}{\|u\|_{L^\infty}} \gtrsim \lambda^{\frac{3-n}{4}}.$$

$\square$

5.3. **Yet another proof.** Another variant of the proof is as follows. We assume again the existence of reflected Brownian motion. The estimate

$$e^{t\Delta_N}u - e^{t\Delta_D}u \leq Ct^{1/2}p_t(x)\|\nabla u\|_{L^\infty}$$

follows from studying the difference between Brownian motion reflected and absorbed at the boundary and using the fact that within t units of time a Brownian motion may travel a distance of up to $\sim t^{1/2}$. Then,

$$\begin{aligned} e^{-\lambda t}\int_D u(x)dx &= \int_D e^{\lambda t\Delta_D}u(x)dx \\ &\geq \int_D e^{\lambda t\Delta_N}u(x) - Ct^{1/2}p_t(x)\|\nabla u\|_{L^\infty(D)}dx \\ &= \int_D u(x)dx - Ct^{1/2}\|\nabla u\|_{L^\infty(D)}\int_D p_t(x)dx. \end{aligned}$$

In the limit $t \to 0^+$,

$$\lim_{t\to 0^+}\frac{1}{\sqrt{t}}\int_D p_t(x)dx \gtrsim \lambda\frac{\|u\|_{L^1}}{\|\nabla u\|_{L^\infty}}$$

and summing over all domains gives with the Sogge-Zelditch inequality the desired result.

5.4. **Geometric structure of nodal sets.** The quantity $p_t(x)$ can be seen as a local measure of the closeness and size of the boundary. It seems extremely natural to conjecture the following.

Conjecture. *Let (M,g) be a compact $C^\infty-$manifold without boundary. There exists a constant $c > 0$ depending only on (M,g) such that if $p_t(x)$ is globally defined with respect to the nodal set of a Laplacian eigenfunction with eigenvalue λ, then*

$$p_{\lambda^{-1}}(x) > c \qquad \text{for all } x \in M.$$

This conjecture, while seeming likely, should be extremely difficult. In particular, combining it with heat content isoperimetry at time $t = \lambda^{-1}$ immediately gives

$$\mathcal{H}^{n-1}\left(\{x \in M : u(x) = 0\}\right) \gtrsim \lambda^{\frac{1}{2}}.$$

5.5. **Heat content, Laplacian eigenvalues and the inradius.** Given a domain $\Omega \in \mathbb{R}^n$, we can define the first eigenvalue of the domain as

$$\lambda_1(\Omega) = \inf_{f\in H_0^1(\Omega)}\frac{\int_\Omega |\nabla f(x)|^2\,dx}{\int_\Omega f(x)^2dx}.$$

An inequality of the form

$$\lambda_1(\Omega) \gtrsim r^{-2},$$

where r is the inradius of the domain is trivial if $n = 1$, true for simply connected domains in $n = 2$ (Hayman's theorem) and false for $n \geq 3$. Indeed, in dimensions $n \geq 3$ it is possible to introduce very thin spikes making the inradius small but having little overall influence on the eigenvalue. However, Lieb [14] in a celebrated paper has shown that Hayman's theorem 'essentially' generalizes to higher dimensions: for any domain $\Omega \in \mathbb{R}^n$, there is a ball B of radius $r \sim \lambda^{-1/2}$ such that $|\Omega \cap B| \sim |B|$ (the theorem also gives a precise relationship between the implicit constants). We conjecture that this phenomenon persists for the heat content.

Conjecture. *Let $\Omega \in \mathbb{R}^n$ be a bounded open set. If $c_1 > 0$ and some $t > 0$*

$$\int_\Omega p_t(x)dx \geq c_1\mathcal{H}^{n-1}(\partial\Omega)\sqrt{t},$$

then there is a ball B of radius $\sqrt{t}$ such that $|B \cap \Omega| \geq c_2|B|$, where c_2 depends only on the dimension and c_1.

Of course, via Lieb's theorem, this would establish a mutual equivalence between the first Laplacian eigenvalue, the size of balls having a large intersection with the domain and the time up to which heat content isoperimetry is sharp ($t = \lambda_1(\Omega)^{-1}$).

References

[1] R. Bass, and E. Hsu, Pathwise uniqueness for reflecting Brownian motion in Euclidean domains. Probab. Theory Related Fields 117 (2000), no. 2, 183–200.

[2] M. van den Berg and P. Gilkey, Heat content asymptotics of a Riemannian manifold with boundary. J. Funct. Anal. 120 (1994), no. 1, 48–71.

[3] J. Brüning, Über Knoten Eigenfunktionen des Laplace-Beltrami Operators, Math. Z. 158 (1978), 15–21.

[4] J. Cheeger, A. Naber and D. Valtorta, Critical sets of elliptic equations, arXiv:1207.4236

[5] T. H. Colding and W. P. Minicozzi II: Lower bounds for nodal sets of eigenfunctions, Comm. Math. Phys. 306 (2011), 777-784.

[6] R. T. Dong, Nodal sets of eigenfunctions on Riemann surfaces. J. Differential Geom. 36 (1992), no.2, 493–506.

[7] H. Donnelly and C. Fefferman, Nodal sets of eigenfunctions on Riemannian manifolds, Invent. Math. 93 (1988), 161–183

[8] H. Donnelly and C. Fefferman, Nodal sets for eigenfunctions of the Laplacian on surfaces, J. Amer. Math. Soc. 3(2) (1990), 332–353.

[9] R. Hardt and L. Simon, Nodal sets for solutions of elliptic equations. J. Differential Geom. 30 (1989), no. 2, 505–522.

[10] W. Hayman, Some bounds for principal frequency. Applicable Anal. 7 (1977/78), no. 3, 247–254.

[11] H. Hezari and C. D. Sogge, A natural lower bound for the size of nodal sets, arXiv:1107.3440, to appear in Analysis and PDE.

[12] H. Hezari and Z. Wang, Lower bounds for volumes of nodal sets: an improvement of a result of Sogge-Zelditch. To appear in AMS Proceedings on Spectral Geometry

[13] Ioannis Karatzas and Steven E. Shreve, Brownian Motion and Stochastic Calculus, Springer, 2008.

[14] E. Lieb, On the lowest eigenvalue of the Laplacian for the intersection of two domains. Invent. Math. 74 (1983), no. 3, 441–448.

[15] D. Mangoubi, Local asymmetry and the inner radius of nodal domains, Comm. Partial Differential Equations 33 (2008), no. 7-9, 1611-1621.

[16] D. Mangoubi, The volume of a local nodal domain, J. Topol. Anal. 2 (2010), no. 2, 259–275.

[17] F. Nazarov, L. Polterovich, and M. Sodin, Sign and area in nodal geometry of Laplace eigenfunctions, Amer. J. Math. 127 (2005), no. 4, 879–910.

[18] C. D. Sogge and S. Zelditch, Lower bounds on the Hausdorff measure of nodal sets, Math. Res. Lett. 18 (2011), 25-37.

[19] C. D. Sogge and S. Zelditch, Addendum to 'Lower bounds on the Hausdorff measure of nodal sets', arXiv:1208.2045.

[20] M. Taylor, Partial differential equations. II. Qualitative studies of linear equations. Applied Mathematical Sciences, 116. Springer-Verlag, New York, 1996.

[21] S. Varadhan, On the behaviour of the fundamental solution of the heat equation with variable coefficients, Comm. Pure Appl. Math., 20 (1967), 431–455.

[22] S.T. Yau, Survey on partial differential equations in differential geometry, Seminar on Differential Geometry, pp. 3–71, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.

Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany